

Perturbative Algebraic Field Theory, and Deformation Quantization

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Dedicated to Sergio Doplicher and John Roberts at the occasion of their 60th anniversary

Abstract. A perturbative formulation of algebraic field theory is presented, both for the classical and for the quantum case, and it is shown that the relation between them may be understood in terms of deformation quantization.

1 Introduction

The algebraic approach to field theory (“Local Quantum Physics”) [22] has deepened and enlarged our understanding of fundamental properties of quantum field theory [8]. As the perhaps most important insight one may mention the theory of superselection sectors [12] which culminated in the work of Sergio Doplicher and John Roberts on a new duality theory for compact groups [13, 14].

On the level of concrete models the algebraic approach was less successful. The unfortunately still insufficient mathematical control on models of quantum field theory in all existing approaches seemed to be a severe obstacle for an application of the framework of algebraic field theory. But recently it was shown that on the level of perturbation theory a quite satisfactory formulation is possible, on the basis of older attempts by Bogoliubov-Shirkov [5], Epstein-Glaser [19], Steinmann [30] and Stora [31]. The main new insight is that the formulation of perturbative quantum field theory in the spirit of local quantum physics admits a complete disentanglement of ultraviolet and infrared problems. One application is the perturbative renormalization on a curved background [6, 7], another a local construction of observables in gauge theories [15]. A third application is a better understanding of the relation between classical and quantum field theory. It can

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be shown that the formalism of deformation quantization can be naturally applied and delivers the loop expansion of perturbation theory [16].

The plan of this paper is as follows: we briefly review the framework of algebraic quantum field theory and describe an analogous formulation of classical field theory. In particular we present a perturbative construction of interacting classical fields.

We then discuss the problem of $*$ -product quantization for free fields. We will show that the Wick quantization plays a distinguished role by allowing an extension to polynomials of fields.

Up to singularities at coinciding points, the expansion of interacting classical fields in terms of iterated retarded Poisson brackets can be transformed into the expansion of the interacting quantum fields in retarded functions just by replacing Poisson brackets by commutators. The fixing of the ambiguities at coinciding points amounts to imposing renormalization conditions.

2 The Algebraic Formulation of Quantum Field Theory

Algebraic quantum field theory essentially relies on 2 principles:

1. Quantum principle: The observables form a noncommutative associative $*$ -algebra with a faithful Hilbert space representation (e.g. a C^* -algebra).
2. Principle of locality: Observables are associated to space time regions.

The second principle allows an interpretation of measurements. If $\mathfrak{A}(\mathcal{O})$ denotes the algebra of all observables which can be measured within the spacetime region \mathcal{O} then the guiding principle of algebraic quantum field theory states that the isotonic,

$$\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathfrak{A}(\mathcal{O}_1) \hookrightarrow \mathfrak{A}(\mathcal{O}_2) ,$$

net $(\mathfrak{A}(\mathcal{O}))_{\mathcal{O}}$ characterizes the theory completely. This principle, originally formulated by Haag and Kastler in [23], has been checked in a huge variety of situations. In particular it emphasizes that the physical interpretation of the theory does not depend on the choice of fields. This was already known at that time for scattering theory (fields from the same Borchers class produce the same S -matrix [2]). It plays a fundamental role in the analysis of superselection sectors [3, 12, 9, 20, 14], and is crucial for the approach of Buchholz and Verch towards an intrinsic renormalization group [10].

If one adopts this principle one gets a convenient notion of equivalence between different theories: two theories are equivalent if and only if their local nets are isomorphic. This notion of equivalence can be applied to an analysis of duality transformations and was recently used in Rehren's work [28] on the relation between theories on $D+1$ -dimensional Anti-de-Sitter spacetime and conformal field theories on D -dimensional Minkowski space originally conjectured by Maldacena in the framework of string theories.

3 The Algebraic Formulation of Classical Field Theory

Let $\mathcal{L}(\varphi, \partial\varphi)$ be the Lagrangian of a scalar field φ leading to the field equation

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \frac{\partial \mathcal{L}}{\partial \varphi} , \tag{3.1}$$

and let \mathcal{C} be the space of smooth solutions with compactly supported Cauchy data. \mathcal{C} may be considered as the classical phase space, and typical observables are the evaluation functionals $F_x(\varphi) := \varphi(x)$. By the usual abuse of notation we write $\varphi(x)$ for F_x . Following Peierls [26], one may define the Poisson bracket of two observables without recourse to a Hamiltonian formulation in the following way.

Let \mathcal{L}_1 be a polynomial in φ (i.e. in F_x) and f be a test function on Minkowski space \mathbb{M} with compact support. Denote by $\varphi_{f\mathcal{L}_1}$ a solution of the field equation (3.1) derived from the Lagrangian $\mathcal{L} + f\mathcal{L}_1$. Then $\varphi_{f\mathcal{L}_1}$ coincides at early times with a solution φ_{in} of the original field equation and at late times with another solution φ_{out} . Provided the Cauchy problem is well posed, we obtain a mapping

$$s(f\mathcal{L}_1) : \begin{cases} \mathcal{C} & \rightarrow & \mathcal{C} \\ \varphi_{\text{in}} & \mapsto & \varphi_{\text{out}} \end{cases}$$

which may be considered as the classical S -matrix. The Poisson bracket between the observables $\mathcal{L}_1(F_x)$ with an arbitrary observable G is now defined by

$$\{\mathcal{L}_1(F_x), G\}(\varphi) \stackrel{\text{def}}{=} \frac{\delta}{\delta f(x)} G(s(f\mathcal{L}_1)^{-1}\varphi)|_{f=0} .$$

If we, for example, take the free Klein-Gordon field and choose $\mathcal{L}_1 = \varphi$, then the interacting field

$$\varphi_{f\mathcal{L}_1} = \varphi - \Delta_{\text{ret}} * f \quad (3.2)$$

solves (3.1), where Δ_{ret} is the retarded solution of the Klein-Gordon equation and $*$ denotes convolution. The incoming field coincides with the original free field φ , and the outgoing field is¹

$$\varphi_{\text{out}} = \varphi - \Delta * f , \quad (3.3)$$

where $\Delta = \Delta_{\text{ret}} - \Delta_{\text{adv}}$ is the commutator function from quantum field theory. For the Poisson brackets of the classical fields $\varphi(x)$ one finds

$$\{\varphi(x), \varphi(y)\} = \Delta(x - y) .$$

One may now start from the Poisson algebra of the free field φ and construct interacting fields perturbatively. So let \mathcal{L}_1 be an arbitrary polynomial and f a test function. The corresponding interacting field is given by the following formal power series in the coupling constant $f \in \mathcal{D}(\mathbb{M})$

$$\begin{aligned} \varphi_{f\mathcal{L}_1}(x) &= \sum_{n \geq 0} \int_{x^0 \geq x_1^0 \geq \dots \geq x_n^0} dx_1 \dots dx_n \\ &\cdot f(x_1) \dots f(x_n) \{ \mathcal{L}_1(x_n), \{ \dots \{ \mathcal{L}_1(x_1), \varphi(x) \} \dots \} \} \end{aligned} \quad (3.4)$$

The integrals are well defined because of the support restrictions on f . One verifies that $\varphi_{f\mathcal{L}_1}(x)$ solves the field equation (3.1), and that for equal times $\varphi_{f\mathcal{L}_1}$ and $\dot{\varphi}_{f\mathcal{L}_1}$ have canonical Poisson brackets [16, 17]. Moreover, when $f \equiv 1$ within the causally complete region \mathcal{O} , the Poisson algebra generated by the interacting field $\varphi_{f\mathcal{L}_1}(x)$ with $x \in \mathcal{O}$ is, up to canonical transformations, independent of f (see section 5). Hence we directly obtain an inductive system of local Poisson algebras of classical observables, which we may consider as the classical theory.

4 Wick quantization

In deformation quantization [1] one studies a family $*_{\hbar}$ of associative products in a given Poisson algebra such that

$$\lim_{\hbar \rightarrow 0} a *_{\hbar} b = ab , \quad \lim_{\hbar \rightarrow 0} \frac{1}{i\hbar} [a, b]_{\hbar} = \{a, b\} .$$

¹Obviously the expression (3.3) fulfills the free field equation and agrees for late times with $\varphi_{f\mathcal{L}_1}$ (3.2).

In the case of free field theory one may choose

$$\varphi(x) *_{\hbar} \varphi(y) = \varphi(x)\varphi(y) + \hbar \Delta'(x-y)$$

where the antisymmetric part of Δ' is determined by the commutator function,

$$\Delta'(x) - \Delta'(-x) = i\Delta(x) . \quad (4.1)$$

For (classical) products of fields one may set

$$\begin{aligned} \varphi(x_1) \cdots \varphi(x_n) *_{\hbar} \varphi(y_1) \cdots \varphi(y_m) = \\ \sum_{\text{contractions}} \prod \hbar \Delta'(x_i - y_k) \prod \varphi(x_j) \varphi(y_l) , \end{aligned} \quad (4.2)$$

with the combinatorics known from Wick's theorem (see (4.3)). There are several possibilities for the choice of Δ' . The most symmetric one is $\Delta' = \frac{i}{2}\Delta$. This choice leads to the Weyl-Moyal quantization. Another choice is $\Delta' = \Delta^+$, where Δ^+ is the positive frequency part of $i\Delta$. Then one gets the so called Wick quantization.

On the level of the algebra generated by the smeared fields $\varphi(f) = \int dx \varphi(x) f(x)$ the different choices lead to isomorphic algebras. Namely, there is a differentiable family T_{\hbar} of linear invertible maps, interpolating between the different $*$ -products, with $T_0 = 1$. To see this we formulate our $*$ -products in terms of generating functionals. We interpret $e^{\varphi(f)}$ as the generating functional for the products of classical fields. Therewith, the formula (4.2) can be written in the form

$$e^{\varphi(f)} *_{\hbar} e^{\varphi(g)} = e^{\varphi(f+g)} e^{\hbar(f, \Delta' g)} \quad (4.3)$$

where $(f, \Delta' g) = \int dx dy f(x) \Delta'(x-y) g(y)$. We set

$$T_{\hbar}(e^{\varphi(f)}) \stackrel{\text{def}}{=} e^{\varphi(f)} e^{\frac{\hbar}{2}(f, (\Delta'' - \Delta') f)} .$$

Then T_{\hbar} interpolates between $*$ -products defined with Δ'' and Δ' , respectively:

$$T_{\hbar}(e^{\varphi(f)}) *_{\hbar}^{\Delta''} T_{\hbar}(e^{\varphi(g)}) = T_{\hbar}(e^{\varphi(f)} *_{\hbar}^{\Delta'} e^{\varphi(g)}),$$

where we use $(f, (\Delta'' - \Delta') g) = (g, (\Delta'' - \Delta') f)$ (which relies on (4.1)).

But if we go beyond this minimal algebra and include also pointwise products of fields (this is necessary for a description of interesting interactions) then the picture changes. We now can accept only functions Δ' for which the products can be defined at coinciding points. We may look, for example, at the products of $\varphi(x)^2$. We obtain

$$\varphi(x)^2 *_{\hbar} \varphi(y)^2 = \varphi(x)^2 \varphi(y)^2 + 2\Delta'(x-y) \varphi(x) \varphi(y) + 2\Delta'(x-y)^2$$

which makes sense only when the square of Δ' can be defined.

A convenient criterion for the existence of products of distributions or, equivalently, for the existence of restrictions of tensor products of distributions to submanifolds of coinciding points can be formulated in terms of the wave front sets [25]. Namely, a distribution can be restricted to a submanifold, if the conormal bundle of the submanifold does not intersect the wave front set of the distribution. The wave front set of the commutator function is

$$\text{WF}(\Delta) = \{(x, k) \in T^*\mathbb{M}, k \neq 0, x, k \text{ lightlike}, k \text{ coparallel to } x\}$$

that of Δ_+ is the positive frequency part $k_0 > 0$ of the wave front set of Δ (see e.g. [27]).

In our example we have to study the conormal bundle N^*D of the diagonal $D = \{(x, x), x \in \mathbb{M}\}$ of \mathbb{M}^2 . It is the orthogonal complement of the tangent bundle of D within $T_D^*\mathbb{M}^2$,

$$N^*D = \{(x, x; k, -k), (x, k) \in T^*\mathbb{M}\} .$$

It certainly cannot intersect the wave front set of $\Delta_+ \otimes \Delta_+$ because of the positive frequency condition whereas there is a nontrivial intersection with the wave front set of $\Delta \otimes \Delta$.

It is now straightforward to see that the $*$ -product with $\Delta' = \Delta_+$ can be extended to the Poisson algebra containing all smeared powers $\varphi^n(f) = \int dx \varphi^n(x) f(x)$ of the free field φ . The mappings T to equivalent $*$ -products are only well defined on this larger Poisson algebra if Δ' differs from Δ_+ by a smooth function. The $*$ -product with $\Delta' = -\Delta_-$, Δ_- being the negative frequency part of $i\Delta$, could also be defined on the larger Poisson algebra. But on the arising associative algebra there is no linear functional ω with $\omega(\mathbf{1}) = 1$ which is nonnegative on $\varphi(f) *_\hbar \varphi(f)$ for all real valued test functions f . Namely, we have

$$\omega(\varphi(f) *_\hbar \varphi(f)) = \omega(\varphi(f)^2) + \hbar(f, \Delta' f) .$$

If f tends to the δ -function, the first term on the right hand side converges whereas the second term tends to $\pm\infty$ for $\Delta' = \pm\Delta_\pm$. We conclude that the algebra in the case of $\Delta' = -\Delta_-$ does not admit a faithful Hilbert space representation with a hermitean field φ .

We observe that, in contrast to quantum mechanics with finitely many degrees of freedom, Wick quantization is distinguished in field theory (see also [11]).

We may now formalize the structure described above. The admissible smearing functions of n -fold products of the free field are symmetrical distributions t_n with compact support and with a wave front set where never all components of the covectors k are contained in the closure of the same component of the lightcone (forward or backward)

$$k \notin \overline{V}_+^n \cup \overline{V}_-^n .$$

The space of all these distributions will be denoted by \mathcal{W}_n . It contains in particular products of a δ -function in the difference variables with a smooth function of compact support (cf. [16]).

The $*$ -product may directly be defined in terms of these smearing functions. Let $\mathcal{W}_0 = \mathbb{C}$ and $\mathcal{W} = \bigoplus_n \mathcal{W}_n$, and let f_n denote the component of $f \in \mathcal{W}$ in \mathcal{W}_n . Then we define an associative product $*_\hbar$ on \mathcal{W} by

$$(t *_\hbar s)_n = \sum_{n+2k=l+m} \hbar^k t_m \otimes_k s_l .$$

Here \otimes_k denotes the k -times, with Δ_+ , contracted tensor product. This is the symmetrical distribution, which is defined on symmetrical test functions $f \in \mathcal{D}(\mathbb{M}^{m+l-2k})$ ($m \geq k, l \geq k$) by

$$\langle t_m \otimes_k s_l, f \rangle = \frac{m!l!}{k!(m-k)!(l-k)!} \langle t_m \otimes s_l, (\hat{\Delta}_+^{\otimes k} \otimes f) \circ \sigma \rangle$$

where $\hat{\Delta}_+(x, y) = \Delta_+(x - y)$ and where σ permutes the components of the coordinates of $(x, y) \in \mathbb{M}^m \times \mathbb{M}^l$ such that

$$\sigma(x_1, \dots, x_m, y_1, \dots, y_l) = (x_1, y_1, \dots, x_k, y_k, x_{k+1}, \dots, x_m, y_{k+1}, \dots, y_l) .$$

It is easy to see that the product is well defined, satisfies the condition on the wave front set and makes \mathcal{W} to an associative algebra. The relation of this abstractly defined algebra with the algebra of smeared Wick products on Fock space is described in the following Theorem (cf. [16]):

Theorem 4.1 *Let ϕ be the mapping from \mathcal{W} into the space of densely defined operators on Fock space*

$$\phi(t) = \sum_n : \varphi^{\otimes n} : (t_n)$$

with the Wick products $: \varphi^{\otimes n} : (x_1, \dots, x_n) =: \varphi(x_1) \cdots \varphi(x_n) :$, $\varphi^{\otimes 0} = \mathbf{1}$. Then ϕ is an algebra homomorphism with the kernel

$$\text{Ke } \phi = \{t, \exists s \in \mathcal{W} \text{ such that } t_n = \sum_i (\partial_i^\nu \partial_{i\nu} + m^2) s_n \quad \forall n\}.$$

So, $((\mathcal{W}/\text{Ke } \phi), *_{\hbar})$ provides a purely algebraic quantization of the given Poisson algebra of the classical free fields, it expresses the algebraic structure of smeared Wick products without using the Fock space. Starting from \mathcal{W} , the Fock representation is induced by the state

$$\omega_0 : \begin{cases} \mathcal{W} & \rightarrow \mathbb{C} \\ t & \mapsto t_0 \end{cases}.$$

via the GNS-construction, in particular it holds

$$\omega_0(t) = (\Omega, \phi(t)\Omega)$$

with Ω the vacuum vector in Fock space (see also [4]).

5 Loop expansion and deformation quantization

Formally, we obtain the interacting quantum field by replacing in the formula for the interacting classical field (3.4) the Poisson bracket by the commutator with respect to the associative product $*_{\hbar}$, divided by $i\hbar$ (cf. [16]). So we do not deform directly the algebra of the perturbative interacting classical fields, instead we deform the underlying Poisson algebra of free fields. For a polynomial interaction, the quantum field becomes a convergent² power series in \hbar ,

$$\varphi_{g\mathcal{L}_1}^{\hbar} = \sum \hbar^n \varphi_{g\mathcal{L}_1}^{(n)}.$$

The n -th term is just the n -loop contribution in an expansion into Feynman diagrams.

Let us introduce the algebra of functions of \hbar with values in the formal power series over $g \in \mathcal{D}(\mathbb{M})$ with coefficients in \mathcal{W} ,

$$\mathcal{V} = \{A : \mathbb{R}_+ \rightarrow \mathcal{W}[[g]]\},$$

with the product

$$(A * B)(\hbar) \stackrel{\text{def}}{=} A(\hbar) *_{\hbar} B(\hbar).$$

The interacting fields generate a subalgebra \mathfrak{A} of \mathcal{V} . \mathfrak{A} may be considered as an algebra of sections of a bundle of algebras \mathfrak{A}_{\hbar} over \mathbb{R}_+ . The elements of order \hbar

²Convergence is meant here in the sense of formal power series in the coupling constant $g \in \mathcal{D}(\mathbb{M})$, in formula: $\forall N \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that $\sum_{k=n}^m \hbar^k \varphi_{g\mathcal{L}_1}^{(k)} = \mathcal{O}(g^N) \quad \forall n, m \geq M$.

generate an ideal \mathfrak{I} within \mathfrak{A} which is also an ideal with respect to the Poisson bracket

$$\{A, B\}(\hbar) \stackrel{\text{def}}{=} \frac{1}{i\hbar} [A, B](\hbar) . \quad (5.1)$$

The quotient $\mathfrak{A}_0 = \mathfrak{A}/\mathfrak{I}$ is the classical Poisson algebra. The powers \mathfrak{I}^n , $n \in \mathbb{N}$, generate the ideals within \mathfrak{A} of the elements of order \hbar^n , and the algebras of observables up to n loops can be defined by

$$\mathfrak{A}_n \stackrel{\text{def}}{=} \mathfrak{A}/\mathfrak{I}^{n+1} \quad (5.2)$$

(cf. [16]). They are noncommutative algebras with an additional Poisson bracket (5.1) which is not defined in terms of the commutator (it comes from the commutator at higher order in \hbar). The projective system

$$\mathfrak{A} \rightarrow \dots \rightarrow \mathfrak{A}_{n+1} \rightarrow \mathfrak{A}_n \rightarrow \dots \rightarrow \mathfrak{A}_0$$

interpolates between the quantum theory \mathfrak{A} and the classical theory \mathfrak{A}_0 .

To make the described reasoning rigorous, one has to treat infrared and ultraviolet problems. The infrared problems are in our approach circumvented, in the first step, by allowing only interactions with compact support in Minkowski space characterized by the choice of a test function g . The ultraviolet problems show up in the difficulty of defining the terms in the perturbative expansion (3.4) of the interacting quantum fields (the retarded products R),

$$A_{g\mathcal{L}_1}(f) = R(\exp_{\otimes}(g\mathcal{L}_1), fA) \stackrel{\text{def}}{=} \sum_n \frac{1}{n!} R((g\mathcal{L}_1)^{\otimes n}, fA) \quad (5.3)$$

(where A is an arbitrary polynomial in φ and its derivatives) as everywhere defined distributions with values in \mathcal{W} . On noncoinciding points they are already defined as symmetrized iterated retarded Poisson brackets (5.1).

In the Bogoliubov-Epstein-Glaser approach [5, 19, 31, 29, 7] one expresses them in terms of time ordered products. The time ordered products can be inductively defined (this procedure is equivalent to ultraviolet renormalization in approaches via regularization) where the ambiguities are governed by the renormalization group. It is somewhat cumbersome to keep track of the \hbar -dependence during this procedure (see [16]). Fortunately, there is an alternative procedure mainly due to Steinmann³ [30] which works directly with the retarded products. Namely, by (3.4) they are causal,

$$R((g\mathcal{L}_1)^{\otimes n}, f\mathcal{L}_2) = 0 \quad \text{for} \quad \text{supp } g \cap (\text{supp } f + \bar{V}_-) = \emptyset , \quad (5.4)$$

symmetrical in the first entry and satisfy the relation

$$\begin{aligned} & R(\exp_{\otimes}(g\mathcal{L}_1) \otimes h\mathcal{L}_2, f\mathcal{L}_3) - R(\exp_{\otimes}(g\mathcal{L}_1) \otimes f\mathcal{L}_3, h\mathcal{L}_2) \\ &= \{R(\exp_{\otimes}(g\mathcal{L}_1), h\mathcal{L}_2), R(\exp_{\otimes}(g\mathcal{L}_1), f\mathcal{L}_3)\}, \end{aligned} \quad (5.5)$$

which goes back to Lehmann-Symanzik-Zimmermann [24] and Glaser-Lehmann-Zimmermann [21]. (For a derivation using Bogoliubov's definition of the interacting fields in terms of time ordered products, see [15].) This relation holds as well in the classical as also in the quantum case (note that suitable factors of \hbar have been

³This book relies on the LSZ-formalism, the retarded functions are the coefficients in Haag's series (which is an expansion of the interacting field in terms of the free incoming fields). However, with some modifications the procedure works also in causal perturbation theory [18], which, in contrast to the LSZ-framework, allows also for massless fields.

included in the definition of the retarded products). It also keeps the same form after performing the quotient with the ideals \mathfrak{I}^n , hence it holds at any order in \hbar .

From this relation we see that, as distributions in $h \otimes f$, the terms of the left hand side are a splitting of the right hand side into a retarded and an advanced part. Therefore, the n -th order term is, in the complement of the origin, uniquely determined by lower order terms, and the extension to everywhere defined distributions follows essentially the same path as in the Bogoliubov-Epstein-Glaser scheme [30, 18].

In a last step we may now remove the restriction to localized interactions, thereby solving the infrared problem on a purely algebraic level. As above let \mathcal{O} be a causally complete region. We consider a change of the interaction outside of \mathcal{O} : $g\mathcal{L}_1 \rightarrow (g\mathcal{L}_1 + f\mathcal{L}_2)$, $\text{supp } f \cap \mathcal{O} = \emptyset$. We decompose $f = f_+ + f_-$ with $\text{supp } f_{\pm} \cap (\mathcal{O} + \bar{V}_{\mp}) = \emptyset$. Now let $h \in \mathcal{D}(\mathcal{O})$. From (5.4) we know $A_{g\mathcal{L}_1+f_+\mathcal{L}_2}(h) = A_{g\mathcal{L}_1}(h)$, and by using (5.3), (5.4) and (5.5) we obtain the differential equation

$$\begin{aligned} \frac{d}{d\epsilon} A_{g\mathcal{L}_1+\epsilon f_-\mathcal{L}_2}(h) &= \sum_n \frac{1}{n!} R((g\mathcal{L}_1 + \epsilon f_-\mathcal{L}_2)^{\otimes n} \otimes f_-\mathcal{L}_2, hA) \\ &= \{\mathcal{L}_{\epsilon}, A_{g\mathcal{L}_1+\epsilon f_-\mathcal{L}_2}(h)\}, \end{aligned}$$

where

$$\mathcal{L}_{\epsilon} = \mathcal{L}_{2(g\mathcal{L}_1+\epsilon f_-\mathcal{L}_2)}(f_-).$$

It is solved by the Dyson series,

$$A_{g\mathcal{L}_1+f\mathcal{L}_2}(h) = \sum_{r=0}^{\infty} \int_0^1 ds_r \int_0^{s_r} ds_{r-1} \dots \int_0^{s_2} ds_1 \{\mathcal{L}_{s_r}, \{\mathcal{L}_{s_{r-1}}, \dots \{\mathcal{L}_{s_1}, A_{g\mathcal{L}_1}(h)\} \dots \}\}, \quad (5.6)$$

which is convergent in the sense of formal power series in the couplings.

We denote by $\mathfrak{A}_{g\mathcal{L}_1}(\mathcal{O})$ the algebra which is generated by the interacting fields $A_{g\mathcal{L}_1}(h)$, $h \in \mathcal{D}(\mathcal{O})$ and A an arbitrary polynomial in φ and its derivatives. By means of (5.6) one verifies that $A_{g\mathcal{L}_1}(h) \rightarrow A_{g\mathcal{L}_1+f\mathcal{L}_2}(h)$ is a canonical (classical field theory) or unitary (quantum field theory) transformation which is independent of A and h . Hence, $\mathfrak{A}_{\mathcal{L}_1}(\mathcal{O})$ may be, as an abstract algebra, identified with $\mathfrak{A}_{g\mathcal{L}_1}(\mathcal{O})$, where $g \in \mathcal{D}$, with $g \equiv 1$ on \mathcal{O} , is arbitrary. Embeddings $\mathfrak{A}_{\mathcal{L}_1}(\mathcal{O}_1) \hookrightarrow \mathfrak{A}_{\mathcal{L}_1}(\mathcal{O})$ for $\mathcal{O}_1 \subset \mathcal{O}$ are inherited from the inclusion $\mathfrak{A}_{g\mathcal{L}_1}(\mathcal{O}_1) \subset \mathfrak{A}_{g\mathcal{L}_1}(\mathcal{O})$ (see [16]) and define a net of algebras which, according to the principles of algebraic quantum field theory, characterizes the theory completely. (For an alternative proof, which relies on the causal factorization of the time ordered products, see [6, 7] for the quantum case and [16] for classical field theory.)

Since the Poisson bracket on \mathcal{V} (5.1) is a power series in \hbar , the canonical (unitary resp.) transformation (5.6) respects the ideal \mathfrak{I} of elements of order \hbar , thus also the nets of algebras up to n loops (5.2) are well defined (cf. [16]).

6 Conclusions and outlook

We have shown that the conceptual frame of algebraic quantum field theory admits a clear formulation of the perturbative construction of QFT. In particular the relation to classical field theory and to deformation quantization, as well as the role of the interpolating theories up to n loops were clarified. One might hope that the latter theories even admit a nonperturbative construction.⁴

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